Feynman diagrams and low-dimensional topology

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We shall describe a program here relating Feynman diagrams, topology of manifolds, homotopical algebra, non-commutative geometry and several kinds of "topological physics".

The text below consists of 3 parts. The first two parts (topological sigma model and Chern-Simons theory) are formally independent and could be read separately. The third part describes the common algebraic background of both theories.

Conventions

Later on we shall use almost all the time the language of super linear algebra, *i.e.*, the word *vector space* often means $\mathbf{Z}/2\mathbf{Z}$-graded vector space and the degree of homogeneous vector v we denote by $\bar{v}$.

In almost all formulas, one can replace $\mathbf{C}$ by any field of characteristic zero.

By *graph* we always mean finite 1-dimensional CW-complex.

For $g \geq 0$ and $n \geq 1$ such that $2g + n > 2$, we denote by $\mathcal{M}_{g,n}$ the coarse moduli space of smooth complex algebraic curves of genus g with n unlabeled punctures.

1 Associative algebras and moduli spaces of algebraic curves: Two constructions

Let V be a differential associative algebra over $\mathbf{C}$ with an even scalar product on it. This means that $V = V_0 \oplus V_1$ is a super vector space endowed with the structure of associative algebra such that

$$V_i \cdot V_j \subset V_{i+j (\mathrm{mod}\, 2)} ,$$

with an odd derivation d, $d^2 = 0$ and a scalar product $(\, , \,)$ on each V_0, V_1 satisfying conditions

(1) $(xy, z) = (x, yz)$,

(2) $(x, dy) = (-1)^{\bar{x}}(dx, y)$.

Denote by $H(V) := \mathrm{Ker}(d)/\mathrm{Im}(d)$ the space of (co)homologies of the complex V. Then $H(V)$ is again an associative super algebra endowed with the induced scalar product.

The first construction associates any differential algebra V with the scalar product (as above) such that

1° $\dim H(V) < \infty$, and

2° the induced scalar product on $H(V)$ is not degenerate,

cohomology classes in $H^{\mathrm{even}}(\mathcal{M}_{g,n}, \mathbf{C})$ for all g, n. In a sense the initial data for this construction is a kind of "non-commutative" homotopy type with Poincaré duality.

The second construction associates any differential algebra V with the scalar product such that

1° $\dim(V) < \infty$,

2° the scalar product on V is not degenerate,

3° $H(V) = 0$,

homology classes in $H_{\mathrm{even}}(\mathcal{M}_{g,n}, \mathbf{C})$ for all g, n.

We will describe here the first construction. The idea arose on my reading [23]. It goes through some generalization of the notion of differential algebra invented by Jim Stasheff [24] many years ago.

A_∞-algebras

By definition, an A_∞-algebra (or, in other terms, *strong homotopy associative* algebra) is a collection $(V, m_1, m_2, \ldots)$ where V is a super vector space and

$m_1 : V \to V$ is an odd map,

$m_2 : V \otimes V \to V$ is an even map,

$m_3 : V \otimes V \otimes V \to V$ is an odd map,

$\ldots$

satisfying the higher associativity condition:

$$\text{for any } n \geq 1,\ x_1, \ldots, x_n \in V_0 \sqcup V_1$$

$$\sum_{\substack{1 \leq k \leq i, 1 \leq j \\ i+j=n+1}} \pm\, m_i(x_1 \otimes x_2 \otimes \ldots \otimes x_{k-1} \otimes m_j(x_k \otimes \ldots \otimes x_{k+j-1}) \otimes x_{k+j} \otimes \ldots \otimes x_n) = 0\,,$$

where the sign is given by the formula

$$\pm = (-1)^{j(\bar{x}_1 + \ldots + \bar{x}_{k-1}) + (j-1)k}\,.$$

Actually the associativity condition is an infinite sequence of bilinear equations on the multiplications m_i.

Examples

$n = 1$: the corresponding equation is $m_1 \circ m_1 = 0$. Hence m_1 is a differential and V is a complex.

$n = 2$: then $m_2 : V \otimes V \to V$ is a morphism of complexes. V is a differential but not necessarily an associative algebra.

$n = 3$: the third equation means that m_2 is associative up to homotopy given by the map m_3.

We see that differential associative algebras are just A_∞-algebras with $m_3 = m_4 = \ldots = 0$. Conversely, for any A_∞-algebra one can construct (applying the bar construction and then the cobar construction) a differential associative algebra which is in a sense homotopy equivalent to the initial A_∞-algebra. The advantage of A_∞-algebras is the possibility of transfering A_∞-structures across quasi-isomorphisms of complexes ("perturbation theory" in differential homological algebra, V.K.A.M. Gugenheim and J. Stasheff [11]). In particular, one can construct a (non-unique!) structure of A_∞-algebra on $H(V)$ which encodes all Massey operations arising on the space of cohomologies.

A_∞-algebras with scalar products

By definition, an A_∞-algebra with a scalar product is a *finite*-dimensional A_∞-algebra V with a fixed nondegenerate even scalar product on V such that for any $n \geq 1$, the $(n+1)$-linear functional $(m_n, \ ) : V^{\otimes(n+1)} \to V$,

$$x_1 \otimes x_2 \otimes \ldots \otimes x_{n+1} \mapsto (m_n(x_1 \otimes \ldots \otimes x_n), x_{n+1})$$

is cyclically (*i.e.*, $\mathbf{Z}/(n+1)\mathbf{Z}$) symmetric (in the graded sense) for n odd and cyclically antisymmetric for n even.

We developed [16] a perturbation theory for the case of algebras with scalar products and obtain higher multiplications on $H(V)$ obeying cyclicity conditions as above for a non-commutative differential algebra V with Poincaré duality (*i.e.*, $H(V)$ is finite-dimensional and the induced scalar product on it is non-degenerate).

We shall construct for any A_∞-algebra with a scalar product an even cohomology class on the moduli space of curves $\mathcal{M}_{g,n}$. It is based on a certain combinatorial model for $\mathcal{M}_{g,n}$ developed by J. Harer, D. Mumford and R. Penner, ([12], [19]).

Stratification of decorated moduli spaces

By definition, the *decorated* moduli space $\mathcal{M}^{\mathrm{dec}}_{g,n}$ is the moduli space of pairs (C, f) where C is a compact connected complex algebraic curve of genus g and $f : C \to \mathbf{R}_{\geq 0}$ is a non-negative function which takes positive values exactly at n points. It is clear that the rational homotopy type of $\mathcal{M}^{\mathrm{dec}}_{g,n}$ is the same as of $\mathcal{M}_{g,n}$. The space $\mathcal{M}^{\mathrm{dec}}_{g,n}$ has a stratification with the strata equal to quotient spaces of euclideean spaces of some dimensions modulo actions of some finite groups.

Define a *ribbon graph* (or a *fatgraph* in other terms) as a graph with fixed cyclic orders on the sets of half-edges attached to each vertex. One can associate an oriented surface with boundary to each ribbon graph by replacing edges by thin oriented rectangles (ribbons) and glueing them together at all vertices according to the chosen cyclic order. A *metric* on the ribbon graph is a map from the set of edges to the set of positive real numbers $\mathbf{R}_{>0}$.

Denote by $\mathcal{R}^{g,n}$ the moduli space of connected ribbon graphs with metric, such that the degrees of all vertices are greater than or equal to 3 and the corresponding surface has genus g and n boundary components.

Theorem 1.1 *$\mathcal{R}^{g,n}$ is isomorphic to $\mathcal{M}_{g,n}^{\mathrm{dec}}$.*

This theorem follows from results of K. Strebel and/or R. Penner (see [25], [19] or an exposition in [12]).

It is clear that $\mathcal{R}^{g,n}$ is stratified by combinatorial types of underlying ribbon graphs. This stratification was used before for the computation of the orbifold Euler characteristic of $\mathcal{M}_{g,n}$ ([13], [20]) and in the proof ([15]) of Witten's conjecture on intersection numbers of standard divisors on the Deligne-Mumford compactification $\overline{\mathcal{M}_{g,n}}$ (see [29]).

The space $\mathcal{R}^{g,n}$ is a non-compact but smooth orbispace (orbifold), so there is a rational Poincaré duality between its cohomology groups and homology groups with closed supports with coefficients in the orientation sheaf. Hence one can compute the rational cohomology of $\mathcal{M}_{g,n}$ using the complex generated as a vector space by equivalence classes of co-oriented strata.

State model on ribbon graphs

Let V be an A_∞-algebra with a scalar product and denote by $v_1, \ldots, v_N$ an orthogonal base of V (here $N = \dim(V)$). We can encode all data in the sequence of cyclically (anti)-symmetric (in the graded sense) tensors with coefficients:

$$T_{i_1,\ldots,i_n} = (m_{n-1}(v_{i_1} \otimes \ldots \otimes v_{i_{n-1}}), v_{i_n})\,, \quad 1 \le i_* \le N \quad n \ge 2\,.$$

Each ribbon graph defines a way to contract indices in the product of copies of these tensors. In other words, V defines a state model on ribbon graphs. The *partition function* $Z(\Gamma)$ of a ribbon graph Γ is the sum over all colorings of edges of Γ into N colors of the products over vertices of Γ of the corresponding coefficients of tensors T_*. For example, the partition function of the skeleton of a tetrahedron is equal (with appropriate corrections of signs) to

$$\sum_{1\le i_1,\ldots,i_6 \le N} T_{i_1 i_2 i_5}\, T_{i_2 i_3 i_6}\, T_{i_1 i_4 i_3}\, T_{i_4 i_5 i_6}\,.$$

Actually the partition function is defined only up to a sign because, for odd n, the tensors T_* are cyclically anti-invariant. One can check that the sign is fixed by a coorientation of the stratum corresponding to Γ.

Theorem 1.2 $\sum_\Gamma Z(\Gamma)\Gamma$ *is a well-defined cochain on* $\mathcal{M}_{g,n}^{\mathrm{dec}}$. *It follows from the higher associativity conditions that this cochain is closed.*

The proof of this theorem is a simple check. In a sense our construction of cohomology classes of $\mathcal{M}_{g,n}$ starting from an A_∞-algebra with a scalar product is analogous to famous constructions of knot invariants (see, for example, [21]). One can check that the resulting cohomology class does not depend on the choices in the construction and is a homotopy invariant of the differential associative algebra with a scalar product such that the induced scalar product on the cohomology space is non-degenerate.

There is a simple series of A_∞-algebras with scalar products. The underlying vector space is an even one-dimensional space $\mathbf{C}^1$. The scalar product is $(1,1) = 1$ and higher multiplications m_k are zero for odd k and arbitrary linear maps $\mathbf{C}^{\otimes k} \to \mathbf{C}$ for even k. We have proved that the linear span of classes obtained from these algebras is the space of all polynomials in Morita-Miller-Mumford classes (the proof will appear elsewhere).

One could expect that it is possible to produce *all* classes of $H^*(\mathcal{M}_{g,n})$ from the above construction. It would be interesting to construct examples of A_∞-algebras giving some new classes.

Also we expect that the structure of an A_∞-algebra with a scalar product appears naturally in the Floer homology of the space of free paths on almost any complex manifold and the corresponding cohomology classes are restrictions of the classes on $\overline{\mathcal{M}_{g,n}}$ arising from the nonlinear sigma-model [29].

At the moment we don't know what kind of algebraic structure gives cohomology classes of the Deligne-Mumford compactification $\overline{\mathcal{M}_{g,n}}$. We hope that further development will lead to a better understanding of topological non-linear sigma-models, mirror symmetry and relations with matrix models of two-dimensional gravity.

Dual construction

We describe here a way to produce *homology* classes on $\mathcal{M}_{g,n}$. The starting ingredient is a finite-dimensional differential associative algebra V with a non-degenerate *odd* scalar product and trivial cohomology.

The right inverse to the scalar product can be considered as an odd element δ of $V \otimes V$. It follows from the compatibility of the scalar product with the differential that δ is closed. From the triviality of $H(V)$ it follows that there exists $\omega \in V \otimes V$ such that $d\omega = \delta$. We can use ω as a "propagator" and tensors $T_{(k)} : V^{\otimes k} \to \mathbf{C}$

$$T(v_1 \otimes \dots v_k) = (v_1 v_2 \dots v_{k-1}, v_k)$$

as "interactions". Again, we obtain a state model on ribbon graphs. Now we will consider the complex which is *dual* to the cochain complex from the previous section.

Theorem 1.3 $\sum_\Gamma Z(\Gamma)\Gamma$ *is a well-defined chain and it is closed. Its homology class does not depend on the choice of* ω.

The proof of this theorem is again a simple check.

One can compute the pairing between cohomology classes arising from A_∞-algebras and homology classes arising from differential associative algebras. The corresponding number will be the sum over all ribbon graphs of the product of two partition functions. One can identify this sum with the decomposition over Feynman diagrams of a finite-dimensional integral (see [3] for an introduction to Feynman rules for mathematicians). Theoretically it gives a way to check non-triviality of classes arising from A_∞-algebras.

2 Perturbative Chern-Simons theory

Before the discussion of more complicated subjects we want to show the reader a simple formula for an invariant of knots in $\mathbf{R}^3$ which arises naturally from perturbative Chern-Simons theory.

Denote by $\omega(x)$ the closed 2-form $\frac{1}{8\pi}\,\epsilon_{ijk}\,\frac{x^i dx^j\wedge dx^k}{|x|^3}$ on $\mathbf{R}^3\backslash\{0\}$ (= standard volume element on S^2 written in homogeneous coordinates). This form appears in the Gauss formula for the linking number of two nonintersecting oriented curves $L_1, L_2 \subset \mathbf{R}^3$:

$$\#\,(L_1, L_2) = \int\limits_{x\in L_1, y\in L_2} \omega(x-y)\,.$$

Theorem 2.1 *For a knot $K : S^1 \hookrightarrow \mathbf{R}^3$ where $S^1 = [0,1]\backslash\{0,1\}$ the following sum*

$$\int\limits_{0<l_1<l_2<l_3<l_4<1} \omega(K(l_1)-K(l_3))\wedge\omega(K(l_2)-K(l_4))$$
$$+\int\limits_{0<l_1<l_2<l_3<1, x\in\mathbf{R}^3\backslash K(S^1)} \omega(K(l_1)-x)\wedge\omega(K(l_2)-x)\wedge\omega(K(l_3)-x) - \frac{1}{24}$$

is an invariant, i.e., does not change when we vary continuously K in the class of embeddings. This invariant is an integer number and it is equal to the second coefficient of the Conway polynomial of the knot $K(S^1)$.

This theorem follows from the study of certain compactifications of domains of integration and general properties of Vassiliev invariants (see below).

Overview of Chern-Simons theory

Let M be a closed oriented 3-dimensional manifold and let G be a compact Lie subgroup of the group of unitary matrices $U(N)$. Denote by A a 1-form on M with values in the Lie algebra $\mathfrak{G}$ of the group G. We can consider A as a connection in the trivial G-bundle over M.

The Chern-Simons functional of A is given by the formula

$$CS(A) = \frac{i}{4\pi} \int_M \mathrm{Tr} \left(A\, dA + \frac{2}{3} A^3 \right) .$$

This functional is invariant under infinitesimal gauge transformations $\delta A = dX + [A, X]$ for $X : M \to \mathfrak{G}$. The quantity $e^{CS(A)}$ is invariant under *all* gauge transformations

$$A \mapsto g^{-1} A g + g^{-1} dg \,, \qquad g : M \to G$$

and also could be defined fro connections in nontrivial bundles.

Formally we can write the integral over the quotient space of the space of all connections in G-bundles over M modulo gauge transformations (for any integer k):

$$Z_k = \int \left(e^{CS(A)} \right)^k \mathcal{D}A \,, \quad k \in \mathbf{Z} \,.$$

Witten [30] in 1988 described a way to "compute" the numbers Z_k (for $k \neq 0$) starting from conformal field theory. These numbers are algebraic numbers belonging to cyclotomic fields. To compute Z_k, one has to decompose M into some simple pieces. Also one needs to know R-matrices for a quantum group with parameter q equal to a root of unity. Then the decomposition of M defines a way to contract indices in some tensor products of R-matrices and some other auxiliary tensors. The result of the computation will be Z_k. For a precise description see [22].

If one believes in the existence of Feynman integrals, then one expects a certain asymptotic behaviour of the complex numbers Z_k for large k. The integral must be concentrated near critical points of CS, *i.e.*, near flat connections. Let us suppose for simplicity that there are only finitely many conjugacy classes of group homomorphisms $\pi_1 M \to G$. Conjecturally, there is an asymptotic formula of the following form:

$$Z_k \sim \sum_{\rho : \pi_1 M \to G} e^{kCS(\rho)} \, \frac{k^{d_\rho}}{R(\rho)} \, \exp\left(\sum_{n=1}^{\infty} a_{n,\rho}\, k^{-n} \right), \quad k \to +\infty$$

where d_ρ, $R(\rho)$ can be computed from the Reidemeister torsion and the dimensions of the cohomologies of the adjoint local system $ad(\rho)$, and $a_{n,\rho}$ are higher term corrections to the Gauss approximation.

This expansion was checked (without higher corrections) for some cases, where explicit formulas for Z_k exists, see [9], [14], and for some more complicated cases using computer simulations in [6].

Recipe for perturbation theory

Here we shall describe a way to define coefficients $a_{n,\rho}$ in the conjectural asymptotic expansion of Z_k. Let us fix a flat connection ∇ in a G-bundle on M and denote by $\mathfrak{g}$ the corresponding local system of Lie algebras (arising from the adjoint representation of G). Suppose for simplicity that $\mathfrak{g}$ is infinitesimally irreducible and rigid,

$$H^0(M,\mathfrak{g}) = H^1(M,\mathfrak{g}) = 0\,.$$

It follows using Poincaré duality that $\mathfrak{g}$ is acyclic.

Connections near the flat connection ∇ can be written as $\nabla + A$ where A is a 1-form with values in $\mathfrak{g}$.

The first step in the standard physical approach to the computation of integrals over spaces of fields modulo gauge transformations is to add Faddeev-Popov ghost and anti-ghost fields. For mathematicians it means that we consider 0 and 2-forms with values in $\mathfrak{g}$ as *odd* (= fermionic) fields.

The second step is to define a so-called gauge fixing. It is enough, for example, to choose a Riemannian metric on M. The gauge fixing gives a propagator which can be considered as a 2-form ω on M^2 with values in $\mathfrak{g}\otimes\mathfrak{g}$. This form is smooth and closed outside the diagonal $M_{\mathrm{diag}} \subset M \times M$. The exterior derivative of ω on all of M^2 must be equal to the product of the delta-form on M_{diag} times the inverse to the scalar product on $\mathfrak{g}$. If M is a Riemannian manifold one can fix ω uniquely as a harmonic form outside M_{diag} (in other words, ω will be a Green form).

The third and the last step is to compute integrals corresponding to Feynman diagrams. Let Γ be a finite nonempty connected graph with all vertices having degree 3 and without simple loops (that is, edges attached at both ends to one vertex). Such a graph has $2n$ vertices and $3n$ edges for some $n \geq 1$. Suppose that all vertices and all edges of Γ are numbered from 1 to $2n$ and to $3n$ respectively, and all edges are oriented. All these data can be encoded into two vectors l_*, r_* of length $3n$, where l_i and r_i are labels of the left and the right vertex respectively attached to the edge i.

the integral corresponding to Γ is given by the formula

$$Z_\Gamma = \int\limits_{(x_1,\dots,x_{2n})\in M^{2n}\setminus \mathrm{diag}} \!\!\!\!\!\!\dots\int \mathrm{Tr}\prod_{i=1}^{3n} \omega(x_{l_i}, x_{r_i})\,,$$

where "Tr" denotes the tensor product over all vertices of skew-symmetric invariant 3-linear functionals on $\mathfrak{g}$:

$$X_1 \otimes X_2 \otimes X_3 \mapsto \mathrm{Tr}\,(X_1X_2X_3 - X_3X_2X_1)\,.$$

Here we use an isomorphism $(\mathfrak{g}^{\otimes 2})^{\otimes 3n} \simeq (\mathfrak{g}^{\otimes 3})^{\otimes 2n}$ arising from the labeling of vertices and edges of Γ.

The n-th term in the perturbation expansion of Z_k must be equal to

$$a_{n,\rho} = \frac{(c)^n}{(2n)!(3n)!}\sum_\Gamma Z_\Gamma$$

for some constant $c \neq 0$.

The factorial factors in the denominator come from counting of numberings of edges and vertices of Γ.

There are two basic problems here:

(1) why the integrals converge, and

(2) why $a_{n,\rho}$ does not depend on the choice of the form ω?

There are two slightly different solutions (1991) of these problems, by Axelrod-Singer [1] and by Kontsevich [16]. The first solution uses Green functions and Riemannian metrics, the second solution is softer in some sense but uses the triviality of the tangent bundle to M. We will describe here our approach.

Compactification of the configuration spaces

Let now M denote a compact manifold of arbitrary dimension. Denote by $\mathrm{Conf}_n(M)$ the space of configurations of n labeled distinct points in M,

$$\mathrm{Conf}_n(M) = M^n \backslash \text{all diagonals}.$$

We shall construct a certain compact manifold with corners (*i.e.*, a space looking locally like a neighbourhood of a point in the closed cube) $\overline{\mathrm{Conf}}_n(M)$ containing $\mathrm{Conf}_n(M)$ as the interior.

The idea is to make all real-analytic blow-ups along all the diagonals $M^k \subset M^n$, $1 \leq k \leq n-1$. For example, $\overline{\mathrm{Conf}}_2(M) = \mathrm{Conf}_2(M) \cup SM$ is a manifold with the boundary $\partial \overline{\mathrm{Conf}}_2(M)$ equal to the total space SM of the spherical bundle associated with the tangent bundle TM.

In general, $\overline{\mathrm{Conf}}_n(M)$ is quite complicated. One can describe it as a set using the following notation:

Let V be a finite-dimensional vector space over $\mathbf{R}$ and m an integer greater than 1. Denote by $C_m(V)$ the quotient space of $\mathrm{Conf}_m(V)$ modulo the action of the semi-direct product of the group V of translations and the group $\mathbf{R}^*_+$ of positive dilatations. The space $C_m(V)$ is smooth because the action is free.

As a set, $\overline{\mathrm{Conf}}_n(M)$ is equal to the disjoint union of some strata. Each stratum is a bundle over a configuration space $\mathrm{Conf}_k(M)$ for some k, $1 \leq k \leq n$. The fiber of this bundle over the point $(x_1, \ldots, x_k) \in \mathrm{Conf}_k(M)$ is a product over some index set A of spaces $C_{m_\alpha}(T_{x_{i_\alpha}})$, $\alpha \in A$. More precisely, strata correspond to abstract oriented forests with endpoints numbered from 1 to n and k connected components (trees). A is the set of vertices which are not endpoints, i_α denotes the subtree containing $\alpha \in A$, m_α is equal to the number of immediate successors of the vertex α.

One can check that the evident forgetful maps $\mathrm{Conf}_n(M) \to \mathrm{Conf}_m(M)$ for $n > m$ can be prolonged to smooth maps $\overline{\mathrm{Conf}}_n(M) \to \overline{\mathrm{Conf}}_m(M)$. Another important fact is the list of codimension one strata in $\overline{\mathrm{Conf}}_n(M)$. They are in one-to-one correspondence with subsets $S \subset \{1, 2, \ldots, n\}$ with the cardinality $\#(S) \geq 2$. Geometrically such a stratum corresponds to the situation when points with labels from S are coming close to each other.

An analogous construction in the context of algebraic geometry was invented by W. Fulton and R. MacPherson [8]. It gives a resolution of singularities of the union of all diagonals in the product of several copies of an algebraic variety.

Let us return to the problem with integrals appearing in the Chern-Simons perturbation theory. First of all, we fix a trivialization of the tangent bundle TM. It gives a decomposition

$$\partial(\overline{\mathrm{Conf}}_2(M)) = SM = S^2 \times M \,.$$

Define ω on $\partial(\overline{\mathrm{Conf}}_2(M))$ as the product of the standard volume element on S^2 by the inverse of the scalar product in $\mathfrak{g}$. It follows from acyclicity of $\mathfrak{g}$ that there exists a closed 2-form ω globally on $\overline{\mathrm{Conf}}_2(M)$. Then the convergence of integrals is evident, because we integrate now a smooth form over a *compact* manifold with corners $\overline{\mathrm{Conf}}_{2n}(M)$.

Topological invariance of the sum of integrals follows from Stokes formula and the following general lemma:

Lemma 2.2 *Let $\omega \in \Omega^{d-1}(S^{d-1}) = \Omega^{d-1}(C_2(\mathbf{R}^d))$ denote any (anti)-symmetric volume element on S^{d-1} (for example, the standard rotation invariant volume element). For any integer $N \geq 3$ and for any two sequences l_i, r_i, $i = 1, \ldots, L$ of integers $l_i \neq r_i$, $1 \leq l_i, r_i \leq N$ and for dimension $d \geq 3$ the integral*

$$\int\limits_{(x_1,\ldots,x_N)\in C_N(\mathbf{R}^d)} \prod_{i=1}^{L} \omega(x_{l_i}, x_{r_i})$$

has value zero.

The proof of this lemma is the following:

The dimension of the integration space $(Nd - d - 1)$ must be equal to the degree of the form $(d-1)L$. It follows that in the graph associated with two vectors l_*, r_* there exists a vertex with degree ≤ 2. If the degree of this vertex is 0 or 1, then the form vanishes. If the degree of this vertex is equal 2, then fixing all vertices except this one we obtain the integral

$$\int\limits_{y\in R^d} \omega(x,y)\,\omega(y,z)$$

considered as $(d-2)$-form on the configuration space of 2 points x, z in $\mathbf{R}^d$. This integral vanishes due to the involution

$$y \mapsto x + z - y$$

changing the sign. $\square$

It follows from the lemma that only the simplest components of the boundary of $\overline{\mathrm{Conf}}_{2n}(M)$ give nonzero contribution to the variation of integrals, namely, they correspond to 2-element subsets in the set of indices. Then the Jacobi identity guarantees cancellation of all boundary terms.

The graph complex

Let us analyse the proof of topological invariance of $a_{n,\rho}$ in the case of trivial ρ. Of course, this case is not realistic because for a nonempty manifold M or nonzero $\mathfrak{g}$, the cohomology group $H^0(M,\mathfrak{g})$ is nontrivial. Nevertheless, imagine such a situation. It is clear that for any graph Γ the corresponding integral Z_Γ is decomposed into the product of two factors,

$$Z_\Gamma = Z_{\Gamma,\mathfrak{G}} \times Z_{\Gamma,M}\,,$$

where the first factor depends only on the Lie algebra $\mathfrak{G}$ with nondegenerate scalar product and the second factor depends only on the manifold M.

The numbers $Z_{\Gamma,\mathfrak{G}}$, $Z_{\Gamma,M}$ depend also on some choices, namely, the choices of the cyclic order of 3-element sets of half-edges attached to each vertex. Both numbers change their sign when one changes the cyclic order at one vertex. We shall call an *orientation* of a 3-valent graph a choice of cyclic orders of all vertices up to an even number of changes. Thus $Z_{\Gamma,\mathfrak{G}}$, $Z_{\Gamma,M}$ are both odd functionals on oriented 3-graphs.

The only property of $Z_{\Gamma,\mathfrak{G}}$ which we use in the proof is a kind of Jacobi identity:

$$Z_{\Gamma_1,\mathfrak{G}} + Z_{\Gamma_2,\mathfrak{G}} + Z_{\Gamma_3,\mathfrak{G}} = 0$$

for triples of 3-graphs Γ_i obtained from a graph Γ' with all vertices except one having degree 3 and the exceptional vertex having degree 4. Graphs Γ_i, $i=1,2,3$ are obtained by "inclusion" of one edge in Γ' instead of the exceptional vertex.

Let us define the graph complex as an abstract vector space over $\mathbf{Q}$ generated by equivalence classes of pairs (Γ,or) where Γ is a connected nonempty graph such that the degrees of all vertices are greater than or equal to 3 and (or) is an orientation of the real vector space $\mathbf{R}^{\{\text{edges of }\Gamma\}} \oplus H^1(\Gamma,\mathbf{R})$. We impose the relation

$$(\Gamma,-\text{or}) = -(\Gamma,\text{or})\,.$$

It follows that $(\Gamma,\text{or})=0$ for every graph Γ containing a simple loop. The reason is that such graphs have automorphisms reversing the orientation in our sense. One can check that for 3-valent graphs we have the same notion of orientation as above.

Define a differential d by the formula (for Γ wihtout simple loops):

$$d(\Gamma,\text{or}) = \sum_{e\in\{\text{edges of }\Gamma\}} (\Gamma/e,\ \text{induced orientation}).$$

Here Γ/e denotes the result of contraction of the edge e, the "induced orientation" is the product of the natural orientation on the codimension-one co-oriented subspace $\mathbf{R}^{\{\text{edges of }\Gamma\}/e} \subset \mathbf{R}^{\{\text{edges of }\Gamma\}}$ and the orientation on $H^1(\Gamma/e,\mathbf{R}) \simeq H^1(\Gamma,\mathbf{R})$. One can easily check that $d^2=0$.

The differential d preserves the dimension of $H^1(\Gamma)$ ("number of loops" in physical language). Thus the graph complex is decomposed into the direct sum of its subcomplexes $C_{n,*}$, $n \geq 1$ consisting of graphs Γ with $\dim(H^1(\Gamma)) = n+1$, the degree $*$ being equal to the number of vertices of Γ. Each subcomplex $C_{n,*}$ is finite-dimensional at each degree and has a finite length.

The homology of the graph complex is called graph homology. It is a challenging problem to compute it. We shall show now that graph homology consists of a kind of characteristic classes for diffeomorphism groups of odd-dimensional manifolds. It is clear that there are at least some nontrivial classes arising from Lie algebras with nondegenerate scalar products.

There exists a version of the graph complex which works for even-dimensional manifolds. One has to replace the vector space $\mathbf{R}^{\{\text{edges of }\Gamma\}} \oplus H^1(\Gamma, \mathbf{R})$ by $\mathbf{R}^{\{\text{edges of }\Gamma\}}$ in the definition of the orientation of a graph Γ.

Application of graph homology

Let M be compact oriented manifold of odd dimension $d \geq 3$ which has the rational homotopy type of a sphere, that is $H^*(M, \mathbf{Q}) = H^*(S^d, \mathbf{Q})$. Remove one point p from M and consider the result as $\mathbf{R}^d$ with the topology changed in a compact subset. Suppose that there exists a trivialization of the tangent bundle of $M \backslash \{p\}$ which coincides with the standard trivialization of $T\mathbf{R}^d$ near infinity.

Denote by $\widetilde{B\mathrm{Diff}}\, M$ a base of a universal smooth bundle with fibers diffeomorphic to M, with a section p and a trivialization of the tangent bundle to the fibers outside p having the same behaviour near p as above. We choose such notation because the homotopy type of this space differs from the classifying space of the diffeomorphism group $\mathrm{Diff}\, M$ by something quite simple.

Using the same technique as for Chern-Simons theory, we can construct differential forms on $\widetilde{B\mathrm{Diff}}\, M$ integrating products of copies of a suitably chosen form ω over the union of configuration spaces of fibers. These forms are labeled by graphs with orientations in our sense. One can check that we obtain a morphism of complexes

$$C_{n,*} \to \Omega^{dn-*}(\widetilde{B\mathrm{Diff}}\, M)\,.$$

Thus graph homology maps to the cohomology of $\widetilde{B\mathrm{Diff}}\, M$.

In the case $d = 3$ any top-degree class in $H_*(C_n)$ gives a zero-degree cohomology class of $\widetilde{B\mathrm{Diff}}\, M$, *i.e.*, just a real number, which is an invariant of M. For example, the simplest non-zero class represented by a graph with 2 vertices and 3 edges connecting both vertices gives an invariant of a homology 3-sphere with spin structure. The formula for this invariant is

$$\int_{C_2(M \backslash \{p\})} \omega^3\,.$$

Theorem 2.3 *This invariant is also invariant under homology trivial spin cobordisms.*

This theorem was proven by Cliff Taubes [26].

Problems

(1) find a geometric construction of these invariants,

(2) prove (or disprove) that graph homology maps to cohomology classes with rational coefficients,

(3) generalize all this to the case of nonhomology spheres with nontrivial tangent bundles and try to incorporate also the group algebra of the fundamental group.

Knots invariants and Vassiliev's theory

Let us use the same notations for Chern-Simons theory as before. For a simple closed oriented curve L in a 3-manifold M and for a finite-dimensional unitary representation ρ of the group G, denote by $W_{\rho,G}$ the gauge-invariant functional (Wilson loop) on the space of connections given by the formula

$$W_{\rho,G}(A) = \text{trace in } \rho \text{ of the monodromy of the connection } A \text{ along } L.$$

Then for any family of loops L_i and representations ρ_i, $i = 1, \ldots, n$ and for any k one can write the Feynman integral

$$Z_k((L_1, \rho_1), \ldots, (L_n, \rho_n)) = \int \prod_{i=1}^{n} W_{\rho_i, L_i} (e^{CS(A)})^k \, \mathcal{D}A \, .$$

Witten [30] gave rules to compute this integral assuming that the curves L_i do not intersect and are also framed (the normal bundles to L_* are trivialized). One can write down terms in the perturbation series for these integrals as well.

Let us restrict ourselves to the case $M = S^3$. Witten showed that these integrals are just versions of Jones invariants of links (see also [21]). Because $\pi_1 S^3 = 1$ there is only one critical point for the Chern-Simons functional, the trivial connection. Thus, integrals corresponding to Feynman diagrams will be the product of algebraic and geometric factors.

We can replace S^3 by the non-compact space $\mathbf{R}^3$ and use as the propagator the Gauss form

$$\omega(x,y) = \frac{1}{8\pi} \, \epsilon_{ijk} \, \frac{(x^i - y^i) \, d(x^j - y^j) \wedge d(x^k - y^k)}{|x-y|^3} \, .$$

The simplest invariant of knots obtained in this way gives a formula which we presented just before the review of Chern-Simons theory (*cf.* Theorem 2.1). It is hard to say who wrote this formula first, probably D. Bar-Natan or E. Guadagnini, M. Martellini and M. Mintchev [10]. It is quite strange that this simple formula (just a generalization of the Gauss formula for the linking number) was invented so late (1988-1989), and not by topologists, but by physicists. Until 1991 (S. Axelrod–I.M. Singer and my work) it was not clear whether higher order integrals converge or not.

It was recognized by D. Bar-Natan and myself that perturbative Chern-Simons theory for knots is closely related with *Vassiliev* invariants (see [27], [28]).

In 1988-1989 V. Vassiliev introduces a class of knot invariants using an idea absolutely independent from Witten's. His approach was the following:

Let us consider the space of embeddings as the complement in the infinite-dimensional vector space of all maps from S^1 to $\mathbf{R}^3$ to the closed subspace of maps with the self-interesting or singular image. We intersect both spaces – the space of knots and its complement, with an appropriate generic family of finite-dimensional vector spaces with increasing dimensions. For example, the spaces of trigonometric polynomial maps of fixed degrees will do. Then it is possible to apply the usual Alexander duality. Of course, we can generalize this to the case of embeddings of an arbitrary manifold into Euclidean space of arbitrary dimension.

The main technical invention of V. Vassiliev is a very simple simplicial resolution of singularities of the space of non-embeddings, which allows one to compute its homology groups with closed support. This technology can be applied to a very broad class of situations, and in good cases it gives a complete description of the weak homotopy type of some function spaces. The case of knots turns out to be borderline. The spectral sequence arising in Vassiliev's approach does not converge well. The zero-degree part of its limit is a certain countable-dimensional subspace in the continuum-dimensional space of all cohomology classes. In particular, the space of knot invariants considered as the 0-degree cohomology group of the space of embeddings contains a countable-dimensional subspace of Vassiliev invariants.

We now give their definition. For any knot $K : S^1 \hookrightarrow \mathbf{R}^3$ and any family of nonintersecting balls

$$B_1, B_2, \ldots, B_n \subset \mathbf{R}^3$$

such that the intersection of any ball with $K(S_1)$ looks from above like one line passing *over* another, one can construct 2^n knots. These knots will be labeled by sequences of $+1$ and -1 of length n. The knot $K_{\epsilon_1,\ldots,\epsilon_n}$ obtained from $K = K(S^1)$ by replacing for all i such that $\epsilon_i = -1$ the part of the knot in the interior of B_i by another standard sample with the first line passing *under* the second line. Of course, $K_{+1,+1,\ldots,+1}$ is the initial knot K.

Let Φ be a knot invariant with values in an abelian group A (for example, $A = \mathbf{Z}, \mathbf{Q}, \mathbf{C}, \ldots$).

Definition. Φ is an invariant of degree less than n if for all K and $B_1, \ldots, B_n$ as above the following equality holds:

$$\sum_{\epsilon_1,\ldots,\epsilon_n} \epsilon_1 \ldots \epsilon_n \, \Phi(K_{\epsilon_1,\ldots,\epsilon_n}) = 0 \, .$$

For $k = 0, 1, \ldots$ denote by V_k the vector space of $\mathbf{Q}$-valued invariants of degree less than $k+1$. The space of all Vassiliev invariants $V = \bigcup_k V_k$ is the space of invariants of finite degree. There is an evident generalization of the notion of Vassiliev invariants to the case of knots and links in arbitrary 3-dimensional manifolds.

We want to mention here several features of these invariants (see [2], [18], [27], [28]):

(1) All terms in the perturbative Chern-Simons theory are invariants of finite order.

(2) The space of invariants of a fixed degree is finite-dimensional, there exists an a priori upper bound on its dimension. Moreover, this space is algorithmically computable. Unfortunately, the only known method to compute this space for a fixed degree takes super-exponential time.

(3) For any Vassiliev's invariant, there exists a polynomial-time algorithm for computing this invariant for arbitrary knots.

(4) It is not hard to prove that if all Vassiliev's invariants for two knots coincide, then their (Alexander, Conway, Jones, Kauffman, HOMFLY, etc.)-polynomial invariants coincide.

(5) The class of Vassiliev invariants coincides with the class of invariants arising from the work of V. Drinfel'd [5] on quasi-Hopf algebras.

One can show that every Vassiliev invariant has a representation as a finite-dimensional integral over some configuration space associated with the pair of manifolds $K(S^1) \subset \mathbf{R}^3$ of a product of copies of the Gauss form. We proposed in [18] also another integral formula for Vassiliev invariants based on [5]. Conjecturally both integral formulas give the same answer.

Vassiliev invariants could be considered as top-degree homology groups of a complex closely related with the graph complex. Exploiting Vassiliev's ideas and perturbative integrals we proved the following fact:

Theorem 2.4 *For $n \geq 4$ let X_n be the space of embeddings of S^1 into R^n. There exist finite-dimensional vector spaces over $\mathbf{Q}$,*

$$V_0^{i,j}, V_1^{i,j}, \quad 0 \leq j \leq i$$

such that for any $n \geq 4$, $k \geq 0$

$$H^k(X_n, \mathbf{Q}) = \bigoplus_{(n-3)i+j=k} V^{i,j}_{n (\mathrm{mod}\, 2)}\,.$$

The proof is organized as follows:

(1) the second term of the Vassiliev spectral sequence gives an estimate above for cohomology groups of X_n,

(2) Feynman diagrams give differential forms on X_n, so produce a lot of cohomology classes,

(3) using combinatorial arguments we show that cohomology of the complex of Feynman diagrams coincide with the second term of the Vassiliev spectral sequence. Thus estimates above and below coincide and the spectral sequence collapses here.

The proof shows that $\bigoplus_i V_1^{i,0}$ is equal to the space V of Vassiliev knot invariants.

Complete exposition of this theorem will appear elsewhere.

3 (Non)commutative symplectic geometry

Here we will show relations between parts 1 and 2 which appear to be very different.

Three infinite-dimensional Lie algebras

Let us define three Lie algebras. The first one, denoted by l_n, is a certain Lie subalgebra of derivations of the free Lie algebra generated by $2n$ elements $p_1, \ldots, p_n, q_1, \ldots, q_n$. By definition, l_n consists of the derivations acting trivially on the element $\sum [p_i, q_i]$.

The second Lie algebra a_n is defined as the Lie algebra of derivations D of the free associative algebra without unit generated by $p_1, \ldots, p_n, q_1, \ldots, q_n$ satisfying the condition $D(\sum (p_i q_i - q_i p_i)) = 0$.

The third Lie algebra c_n is the Lie algebra of polynomials $F \in \mathbf{Q}\,[p_1, \ldots, p_n, q_1, \ldots, q_n]$ such that $F(0) = F'(0) = 0$, with respect to the usual Poisson bracket

$$\{F, G\} = \sum \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right) .$$

One can define c_n also as the Lie algebra of derivations of a free polynomial algebra $\mathbf{Q}[p_*, q_*]$ preserving the form $\sum dp_i \wedge dq_i$ and the codimension one ideal $(p_1, \ldots, p_n, q_1, \ldots, q_n)$.

Our aim is computation of the stable homology (with trivial coefficients) of these Lie algebras. The spirit of the (quite simple) computations is somewhere between Gelfand-Fuks computations (see [7]) and cyclic homology.

If we denote by h_n one of these three series of algebras, then we have a sequence of natural embeddings $h_1 \subset h_2 \subset \ldots \subset h_\infty$ where the last algebra corresponds to the case of countable infinite number of generators. Of course, $H_*(h_\infty) = \varinjlim H_*(h_n)$.

For the limit algebras h_∞ we have a structure of Hopf algebra on its homology (as is usual in K-theory). The multiplication comes from the homomorphism $h_\infty \oplus h_\infty \to h_\infty$ ($\infty + \infty = \infty$ in set theory) and the comultiplication is dual to the multiplication in cohomology.

This Hopf algebra is commutative and cocommutative. Thus $H_*(h_\infty)$ is a free polynomial algebra (in the $\mathbf{Z}/2\mathbf{Z}$-graded sense) generated by the subspace $PH_*(h_\infty)$ of primitive elements.

In all three cases we have an evident subalgebra $sp(2n) \subset h_n$ consisting of linear derivations. The primitive homology of $sp(2\infty)$ is well-known:

$$PH_k(sp(2\infty), \mathbf{Q}) = \begin{Bmatrix} \mathbf{Q} & k = 3(\operatorname{mod} 4) \\ 0 & k \neq 3(\operatorname{mod} 4) \end{Bmatrix} .$$

Now we can state our main result:

Theorem 3.1 *For all three cases $PH_k(h_\infty)$ is equal to the direct sum of $PH_k(sp(\infty))$ and*

(1) (for the case l_∞)

$$\bigoplus_{n \geq 2} H^{2n-2-k}(\operatorname{OutFree}(n), \mathbf{Q}) ,$$

where OutFree (n) *denotes the group of outer automorphisms of a free group with n generators,*

(2) (for the case a_∞)

$$\bigoplus_{n>0, 2-2g-n<0} H^{4g-4+2n-k}(\mathcal{M}_{g,n}, \mathbf{Q}) ,$$

(3) (for the case c_∞)

$$\bigoplus_{n \geq 2} (\text{graph homology})_k^{(n)} .$$

The gradings on the homology groups arising from the natural grading on h_∞ are equal to $(2n-2)$, $(4g-4+2m)$ and $(2n-2)$, respectively.

Idea of the proof of Theorem 3.1

Recall that our Lie algebras h_n are $\mathbf{Z}_{\geq 0}$-graded. Thus the standard chain complex $\bigwedge^*(h_n)$ is graded. We consider the case when n is much larger than the grading degree.

It is well known that every Lie algebra acts (through the adjoint representation) trivially on its homology. The algebra $sp(2n) \subset h_n$ acts reductively on $\bigwedge^*(h_n)$. Hence the chain complex is canonically quasi-isomorphic to the subcomplex of $sp(2n)$-invariants. It is easy to see that this subcomplex stabilizes when $n \to \infty$.

We claim that for the case c_∞ the subcomplex of invariants is a version of the graph complex constructed from all not necessarily connected graphs with degrees of all vertices ≥ 2.

The underlying vector space of the Lie algebra c_n as a representation of $sp(2n)$ is equal to

$$\bigoplus_{j \geq 2} S^j(V) ,$$

where $V = \mathbf{Q}\langle p_1, \ldots, p_n, q_1, \ldots, q_n\rangle$ is the defining $2n$-dimensional representation of $sp(2n)$.

Thus our chain complex as a representation of $sp(2n)$ is equal to the sum

$$\bigoplus_{k_2 \geq 0, k_3 \geq 0, \ldots} (\wedge^{k_2}(S^2(V)) \otimes \wedge^{k_3}(S^3(V)) \otimes \ldots).$$

Every summand is a space of tensors on V satisfying some symmetry conditions.

From the main theorem of invariant theory, it follows that invariant tensors are obtained by contraction of indices. Then one can easily identify all possible ways to contract indices with appropriate graphs.

Consider the subcomplex consisting of all non-empty connected graphs. It contains as a direct summand the complex of graphs with degrees of all vertices equal 2. This part gives the primitive stable homology of $sp(2\infty)$. The rest is quasi-isomorphic to our standard graph complex using some spectral sequence arguments. This proves Theorem 3.1 for the commutative case.

In the associative and the Lie case we need some explicit description of the corresponding Lie algebras:

Theorem 3.2 *Let V denote a standard symplectic vector space $\mathbf{Q}^{2n}$.*

(1) a_n is isomorphic to $\bigoplus_{k\geq 2}(V^{\otimes k})^{\mathbf{Z}/k\mathbf{Z}}$ as $sp(2n)$-module,

(2) l_n is isomorphic to $\bigoplus_{k\geq 2}(V^{\otimes k}\otimes L_k)^{\Sigma_k}$ where L_k is the $(k-2)!$-dimensional representation of the symmetric group Σ_k with character

$$\chi(1^k) = (k-2)!,\ \chi(1^1 a^b) = (b-1)!\, a^{b-1}\, \mu(a),\ \chi(a^b) = -(b-1)!\, a^{b-1}\, \mu(a)$$

and $\chi(*) = 0$ for all other conjugacy classes of permutations (μ is the Möbius function).

For the proof of this theorem we developed a language of non-commutative geometry. In a sense a_n and l_n are Lie algebras of hamiltonian vector fields in non-commutative associative geometry and Lie geometry respectively (see [17]).

In the associative case, the main theorem of invariant theory gives a version of the graph complex consisting of ribbon graphs. Thus we can use stratification of the moduli space of curves (Theorem 1.1) and obtain Theorem 3.2 for a_∞.

Denote by $\mathcal{G}^{(n)}$ for $n \geq 2$ the set of equivalence classes of pairs $(\Gamma, \mathrm{metric})$ where Γ is a nonempty connected graph with Euler characteristic equal $(1-n)$ and degrees of all vertices greater than or equal to 3, (metric) is a map from the set of edges to the set of positive real numbers $\mathbf{R}_{>0}$. One can introduce a topology on $\mathcal{G}^{(n)}$ using the Hausdorff distance between metrized spaces associated in the evident way with pairs $(\Gamma, \mathrm{metric})$. It is better to consider $\mathcal{G}^{(n)}$ not as an ordinary space, but as an orbispace (*i.e.*, don't forget automorphism groups). We mention here that $\mathcal{G}^{(n)}$ is a non-compact and non-smooth locally polyhedral space. It has a finite stratification by combinatorial types of graphs with strata equal to some quotient spaces of Euclidean spaces modulo actions of finite groups.

A fundamental fact about the topology of $\mathcal{G}^{(n)}$ is the following theorem of M. Culler and K. Vogtmann (see [4]):

Theorem 3.3 *$\mathcal{G}^{(n)}$ is a classifying space of the group* $\mathrm{OutFree}\,(n)$ *of outer automorphisms of a free group with n generators.*

We can construct a finite cell-complex homotopy equivalent to $B\,\mathrm{OutFree}\,(n)$ passing from the natural stratification of $\mathcal{G}^{(n)}$ to its barycentric subdivision. The corresponding cochain complex carries a filtration by graphs (by the minimal graph corresponding to the strata attached to the cell). Computations show that the spectral sequence associated with this filtration collapses at the second term to the Lie version of the graph complex. □

Graph homology also has an interpretation in terms of the moduli space of graphs:

$$H_k(C_{n,*}) = H^{\text{closed}}_{k+n-1}(\mathcal{G}^{(n)}, \epsilon)\,,$$

where ϵ is one-dimensional local system given by orientations of graphs.

Strong homotopy algebras and their characteristic classes

A structure of A_∞-algebra (= *strong homotopy associative* algebra) on a super vector space V is equivalent to an odd derivation d of the free associative super algebra without unit generated by $V^* \otimes \mathbf{C}^{0|1}$ satisfying the equation $[d,d]=0$. Also an A_∞-algebra with a scalar product is the same as a homomorphism of Lie superalgebras

$$\mathbf{C}^{0|1} \to a_{k|l} \otimes \mathbf{C}\,.$$

Here $a_{k|l}$ denotes the Lie superalgebra constructed from the symplectic super vector space $\mathbf{Q}^{2k|l}$ in the same way as a_n is constructed from $\mathbf{Q}^{2n}$ (see Theorem 3.2).

One can define in an analogous way a *strong homotopy Lie* algebra with a scalar product as a homomorphism $\mathbf{C}^{0|1} \to c_* \otimes \mathbf{C}$ and a *strong homotopy associative commutative* algebra with a scalar product as a homomorphism $\mathbf{C}^{0|1} \to l_* \otimes \mathbf{C}$. Mention here that the definition for strong homotopy Lie algebras uses the commutative version of hamiltonian vector fields and vice versa. In many senses the associative case is self-dual.

The homomorphism $\mathbf{C}^{0|1} \to h_* \otimes \mathbf{C}$ composed with the inclusion $h_* \hookrightarrow h_{\infty|\infty}$ gives a map

$$\mathbf{C}^1 = H_{2k}(\mathbf{C}^{0|1}) \to H_{2k}(h_{\infty|\infty}) \otimes \mathbf{C} = H_{2k}(h_\infty) \otimes \mathbf{C}$$

for all positive integer k.

Thus any strong homotopy algebra with a scalar product produces classes in the corresponding version of the graph complex. As in the associative case, there are two constructions starting from differential algebras with scalar products giving characteristic classes.

Perturbative Chern-Simons theory near the trivial representation can be reformulated as the pairing between characteristic class arising from a finite-dimensional Lie algebra with a scalar product (considered as a differential algebra with $d = 0$) and the class arising from a differential commutative algebra (= the de Rham complex of a 3-manifold) with a non-degenerate odd scalar product and trivial cohomology.

Topological applications

There are two obvious functors

$$\{\text{commutative algebras}\} \to \{\text{associative algebras}\} \to \{\text{Lie algebras}\} .$$

They give morphisms of the corresponding Lie algebras and maps between their homology groups.

Thus we obtain maps

$$H^*(\mathcal{M}_{g,n}) \to H^*(\widetilde{B\mathrm{Diff}}\, M, \mathbf{R})$$

for any $g, n \geq 1$ and for any 3-dimensional rational homology sphere M. In particular, at degree zero, we obtain a number which is an invariant associated with the topological type of an oriented surface with boundary and of the 3-manifold M. Also one can increase the dimension of M and make some shift in degrees.

Witten [31] proposed an "explanation" of this map. Let us choose a generic Riemannian metric on M. Then the tangent space TM can be endowed with an almost complex structure. Moreover, it is formally a 3-dimensional almost Calabi-Yau manifold. It follows from the index theorem that (the virtual) dimension of the space of holomorphic curves in TM with boundary at the zero section is equal zero. The "number" of such curves of genus g with n boundary components must be the same number which we obtained using graphs. Also, if one considers a cycle in $B\mathrm{Diff}\, M$ then one obtains a cycle in $\mathcal{M}_{g,n}$.

Unfortunately, it seems at the moment that this construction will not work well and must be improved.

Let us now consider the Lie algebra l_∞. It is clear that it is a Lie analogue of the Teichmüller group $T_{g,1}$ for large genus $g \to \infty$. One can make this analogy more precise and construct (using Theorem 3.1) maps

$$H_k(\mathrm{OutFree}\,(n), \mathbf{Q}) \to H^{2n-2-k}(\mathcal{M}_\infty, \mathbf{Q}) .$$

We don't know what is the geometric or "physical" meaning of these maps.

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